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Minimal uncertainty and maximal information for quantum position and momentum

Paul Busch† and Pekka J Lahti‡

† Institute for Theoretical Physics, University of Cologne, West Germany
‡ Department of Physical Sciences, University of Turku, Finland

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Abstract. In view of the recently proposed entropic uncertainty relations several characterisations of 'maximal information' and 'minimal uncertainty' are compared and applied to position and momentum. It is argued that a sufficiently refined entropic uncertainty relation might require the general concept of POV observables as used in stochastic quantum mechanics.

1. Introduction

In recent years several authors have presented formulations of quantum uncertainty relations (UR) in terms of entropy or information ('entropic UR'). Their aim was to replace the usual 'standard UR' (for standard deviations)

$$(\Delta_\varphi A)^2(\Delta_\varphi B)^2 \geqslant \tfrac{1}{4}|\langle[A, B]_-\rangle_\varphi|^2+\tfrac{1}{4}|\langle\{A, B\}_+\rangle_\varphi - 2\langle A\rangle_\varphi\langle B\rangle_\varphi|^2 \tag{1}$$

by an inequality of the form

$$I_\varphi(A)+I_\varphi(B)\leqslant I^0(A, B) \tag{2}$$

as more adequate expression for the 'uncertainty principle' [1-6]. It is the purpose of this paper to show that this viewpoint needs further elaboration and refinement. It appears desirable to establish not just the supremum of (2) as an overall measure of uncertainty but rather a more refined state-dependent bound

$$I_\varphi(A)+I_\varphi(B)\leqslant I^0_\varphi(A, B). \tag{3}$$

This is because, firstly, a rigorous comparison between the entropic and the standard UR must include an account of their logical relationship. Secondly, we shall encounter examples of non-commuting observables possessing trivial maximum $I^0(A, B)=0$ in (2) for which the search of 'local' maxima $I^0_\varphi(A, B)$ turns out to be highly useful.

Of course, we have not been able to derive a general relation of form (3) but instead shall compare various characterisations of 'maximal information' and point out their connection with 'minimal uncertainty' (§ 2). These notions will be applied to spectral projections of position and momentum where they provide a completion and information theoretic interpretation of previous results by one of the authors [7] (§ 3). The relation between the standard and the entropic formulations can be studied up to now only for position and momentum UR. It has been shown that the 'Everett (entropic) UR' (of form (2)) *implies* the famous Heisenberg UR $\Delta q\Delta p\geqslant \hbar/2$ [1, 2, 4, 8]. One can take a further step towards a comparison of (1) and (3) for position and momentum in the framework of stochastic quantum mechanics (§ 4).

2. Maximal information and minimal uncertainty

In the following considerations we shall describe observables as POV (positive-operator-valued) measures (also called effect-valued, or semispectral measures) as they occur, for example, in quantum optics, quantum theory of open systems, stochastic quantum mechanics and other fields of quantum physics. That is, an observable will be a positive, σ additive, normalised map from some Boolean algebra into the set of Hilbert space 'effects', $A:\Sigma \to \mathscr{E}(\mathscr{H})$. Effects are positive self-adjoint bounded operators with spectrum within $[0, 1]$ $(0 \leqslant E \leqslant 1)$ so that their expectation values can be interpreted as probabilities. Since the projections $\mathscr{P}(\mathscr{H}) \subseteq \mathscr{E}(\mathscr{H})$ are also special effects, the usual spectral measures are subsumed under the above generalised observable concept. If projections are interpreted as properties of quantum systems it can be shown that effects in general represent a kind of 'unsharp' property [9].

The Deutsch–Partovi [5, 6] entropic UR can immediately be generalised to effect-valued observables. Let A, B be a pair of observables, $\mathscr{E} = \{E^i\}_{i\in I} \subseteq \mathscr{R}(A)$, $\mathscr{F} = \{F_j\}_{j\in J} \subseteq \mathscr{R}(B)$ be countable subsets of the ranges of A and B such that $\Sigma_i E^i = \Sigma_j F^j = 1$. We further denote $E_\varphi = \langle\varphi|E|\varphi\rangle$ for any operator E. Then the information in state φ with respect to the partition $\mathscr{E}(\mathscr{F})$ of A (B) is

$$I_\varphi^{\mathscr{E}}(A) = \Sigma_i E_\varphi^i \ln(E_\varphi^i) \qquad I_\varphi^{\mathscr{F}}(B) = \Sigma_j F_\varphi^j \ln(F_\varphi^j) \tag{4}$$

and the sum can be written as

$$I_\varphi^{\mathscr{E}}(A) + I_\varphi^{\mathscr{F}}(B) = \Sigma_{ij} E_\varphi^i F_\varphi^j \ln(E_\varphi^i F_\varphi^j).$$

Now the following inequalities hold true not only for projections but for the arbitrary bounded self-adjoint operators:

$$\begin{aligned} \|E^i + F^j\|^2 &\geqslant \langle\varphi|(E^i + F^j)^2|\varphi\rangle \geqslant \langle\varphi|E^i + F^j|\varphi\rangle^2 \\ &= (E_\varphi^i)^2 + (F_\varphi^j)^2 + 2E_\varphi^i F_\varphi^j \geqslant 4E_\varphi^i F_\varphi^j \end{aligned} \tag{5}$$

from which we obtain the Deutsch–Partovi UR

$$I_\varphi^{\mathscr{E}}(A) + I_\varphi^{\mathscr{F}}(B) \leqslant 2\ln\left(\sup_{ij} \frac{\|E^i + F^j\|}{2}\right) \tag{6}$$

as a realisation of (2). One may also give a sharper state-dependent bound in the sense of (3):

$$I_\varphi^{\mathscr{E}}(A) + I_\varphi^{\mathscr{F}}(B) \leqslant 2\ln\left(\sup_{ij} \frac{E_\varphi^i + F_\varphi^j}{2}\right). \tag{7}$$

In the following we restrict ourselves mainly to 'simple' observables (defined on the smallest non-trivial Boolean algebra $\Sigma = \{0, a, a', 1\}$): that is, we are interested in information with respect to single effects E:

$$I_\varphi(E) = E_\varphi \ln(E_\varphi) + E'_\varphi \ln(E'_\varphi) \qquad (E' = I - E). \tag{8}$$

Non-commutativity or incompatibility of (unsharp) properties E and F will, in general, exclude the possibility of measuring or preparing both of them simultaneously. In particular, if $E = E^Q(X)$, $F = E^P(Y)$ are position and momentum spectral projections associated with bounded measurable sets X, Y, then

$$E^Q(X) \wedge E^P(Y) = 0 \tag{9}$$

holds or, equivalently,

$$\begin{aligned}\langle\varphi|E^Q(X)|\varphi\rangle=1&\Rightarrow\langle\varphi|E^P(Y)|\varphi\rangle<1\\ \langle\varphi|E^P(Y)|\varphi\rangle=1&\Rightarrow\langle\varphi|E^Q(X)|\varphi\rangle<1.\end{aligned}\tag{10}$$

Thus 'certain' position and momentum determinations exclude each other, and the question arises as to what 'degree of certainty' they can be 'known' simultaneously. To answer this question, one may take any reasonable characterisation of maximal joint knowledge, or joint information. For example, statement (10) can be put into the following equivalent forms:

$$E_\varphi+F_\varphi<2 \qquad E_\varphi\cdot F_\varphi<1. \tag{11}$$

In [7] it has been shown that the first expression $E_\varphi+F_\varphi$ can be maximised, and an explicit construction procedure for the corresponding 'state of maximal information' has been given (cf § 3). Here we shall study the question of maxima for this quantity as well as for $E_\varphi\cdot F_\varphi$ and for $I_\varphi(E)+I_\varphi(F)$ for an arbitrary pair of effects E and F. In particular, we shall show that each quantity can be maximal only if there exist states which lead to minimal uncertainty product in (1). Furthermore, in the case of projections the maxima of $I_\varphi(E)+I_\varphi(F)$ (if they exist) coincide with those of one of the quantities $E^\nu_\varphi+F^\mu_\varphi$ and $E^\nu_\varphi\cdot F^\mu_\varphi$ ($E^\nu\in\{E, E'\}$, $F^\mu\in\{F, F'\}$).

For maximal $E_\varphi+F_\varphi$ the variation of

$$\langle\varphi|E|\varphi\rangle+\langle\varphi|E|\varphi\rangle-\lambda\langle\varphi|\varphi\rangle$$

must vanish which implies the following equation:

$$(E+F)|\varphi\rangle=(E_\varphi+F_\varphi)|\varphi\rangle. \tag{12}$$

Multiplying with E or with F and taking the expectation yields

$$(\Delta_\varphi E)^2=(\Delta_\varphi F)^2=-(\langle\varphi|EF|\varphi\rangle-E_\varphi\cdot F_\varphi)=-\mathrm{cov}_\varphi(E,F) \tag{13}$$

which leads to a minimal uncertainty relation (1)

$$(\Delta_\varphi E)^2\cdot(\Delta_\varphi F)^2=[\mathrm{cov}_\varphi(E,F)]^2. \tag{14}$$

(Note that the second term in (1) represents the covariance, or correlation, $\mathrm{cov}_\varphi(A,B):=\frac{1}{2}\langle\varphi|AB+BA|\varphi\rangle-\langle\varphi|A|\varphi\rangle\langle\varphi|B|\varphi\rangle$ between the observables A and B in the state φ.) Similarly, maximising the product $E_\varphi\cdot F_\varphi$ gives

$$(F_\varphi E+E_\varphi F)|\varphi\rangle=2E_\varphi\cdot F_\varphi|\varphi\rangle \tag{15}$$

and

$$(\Delta_\varphi E)^2F_\varphi^2=(\Delta_\varphi F)^2E_\varphi^2=-E_\varphi F_\varphi\,\mathrm{cov}_\varphi(E,F) \tag{16}$$

which again leads to (14), ($E_\varphi\neq 0\neq F_\varphi$).

Finally, maximal information sum $I_\varphi(E)+I_\varphi(F)$ will be realised in states satisfying

$$(\ln E_\varphi-\ln E'_\varphi)(E-E_\varphi)|\varphi\rangle+(\ln F_\varphi-\ln F'_\varphi)(F-F_\varphi)|\varphi\rangle=0. \tag{17}$$

Generally this equation contains all stationary points, e.g. the minimum $E_\varphi=E'_\varphi=F_\varphi=F'_\varphi=\frac{1}{2}$, or the joint eigenstates. Since we are looking for states of maximal information with respect to *positive* outcomes for E, F we shall assume $E_\varphi>\frac{1}{2}$ and $F_\varphi>\frac{1}{2}$. Then (17) implies

$$(\alpha E+F)|\varphi\rangle=(\alpha E_\varphi+F_\varphi)|\varphi\rangle \qquad \alpha=\frac{\ln(E_\varphi/E'_\varphi)}{\ln(F_\varphi/F'_\varphi)}\geqslant 0 \tag{18}$$

and

$$\alpha(\Delta_\varphi E)^2 = \frac{1}{\alpha}(\Delta_\varphi F)^2 = -\mathrm{cov}_\varphi(E, F) \tag{19}$$

which again gives rise to the minimal uncertainty product (14). We note that due to (19) α can be written as $\alpha = \Delta_\varphi F/\Delta_\varphi E$ so that (18) reads equally

$$\begin{gathered}[(\Delta_\varphi F)E + (\Delta_\varphi E)F]|\varphi\rangle = [(\Delta_\varphi F)E_\varphi + (\Delta_\varphi E)F_\varphi]|\varphi\rangle \\ \left(\frac{E_\varphi}{1-E_\varphi}\right)^{\Delta_\varphi E} = \left(\frac{F_\varphi}{1-F_\varphi}\right)^{\Delta_\varphi F}.\end{gathered} \tag{20}$$

We have thus shown that all the three notions of maximal information are consistent insofar as they imply minimal uncertainty product. However, it will depend on the spectra of E and F whether the solutions of the above sets of equations coincide. This coincidence will occur if $E_\varphi = F_\varphi$ which in turn will always hold for *projections* $E = E^2$, $F = F^2$—as can be seen readily from equations (13), (16) and (20); the only alternative solution $E_\varphi = F'_\varphi$ is excluded by the assumption $E_\varphi > \frac{1}{2}$, $F_\varphi > \frac{1}{2}$. This result follows from the fact that for projections one has $(\Delta_\varphi E)^2 = E_\varphi - E_\varphi^2$ so that all the above conditions can be expressed in terms of the variables E_φ, F_φ. It follows further that equations (15) and (20) for φ collapse into (12) so that indeed all the three approaches lead to identical characterisations of maximal information.

Before turning to the discussion of position and momentum we should mention as an example of the above calculations the case of two orthogonal spin components. Mamojka [3] has shown that

$$-2\ln(2) \leq I_\varphi(s_x) + I_\varphi(s_z) \leq -\ln(2) \tag{21}$$

where both bounds are reached by eigenstates of s_x or s_z (for maximal information) and of s_y (for minimal information). Inequality (21) shows that the missing information ranges between one and two bits. Mamojka also noted that maximal information coincides with minimal uncertainty, an observation which we have seen to be true for arbitrary pairs of simple observables. Finally we remark that the absolute maxima of $E_\varphi \cdot F_\varphi$ and $E_\varphi + F_\varphi$ can be determined in an elementary way without resorting to variational methods (see appendix 1).

3. The position–momentum example

Throughout this section E and F denote position and momentum spectral projections, respectively: $E = E^Q(X)$, $F = E^P(Y)$. In [7] the sum of probabilities $E_\varphi + F_\varphi$ has been shown to be maximal in the state $\varphi = \varphi_{mi}$:

$$|\varphi_{mi}\rangle = \left(\frac{1+a_0}{2a_0^2}\right)^{1/2} E|g_0\rangle + \left(\frac{1-a_0}{2(1-a_0^2)}\right)^{1/2} E'|g_0\rangle \tag{22}$$

provided that X, Y are bounded measurable sets. Here a_0^2 is the maximal eigenvalue of the compact operator FEF and g_0 is the corresponding eigenvector satisfying

$$FEF|g_0\rangle = a_0^2|g_0\rangle \qquad F|g_0\rangle = |g_0\rangle \qquad \|g_0\|_2 = 1. \tag{23}$$

From the considerations of the preceding section it is clear that φ_{mi} must be an eigenstate of $E + F$. This can also be seen directly in the following way. Introduce

$$|f_0\rangle = a_0^{-1}E|g_0\rangle \qquad \|f_0\|_2^2 = a_0^{-2}\langle g_0|FEF|g_0\rangle = 1 \qquad E|f_0\rangle = |f_0\rangle \tag{24}$$

then due to (23) we have

$$EFE|f_0\rangle = a_0^2|f_0\rangle \qquad |g_0\rangle = a_0^{-1}F|f_0\rangle \tag{25}$$

and φ_{mi} can be written in the symmetric form

$$|\varphi_{mi}\rangle = \frac{|f_0\rangle + |g_0\rangle}{[2(1+a_0)]^{1/2}}. \tag{26}$$

Furthermore, $(E+F)(f_0+g_0) = (1+a_0)(f_0+g_0)$ which proves

$$(E+F)|\varphi_{mi}\rangle = (1+a_0)|\varphi_{mi}\rangle \qquad E_{\varphi_{mi}} + F_{\varphi_{mi}} = 1 + a_0. \tag{27}$$

(An especially short proof of (22)-(27) is given in appendix 2 which avoids any reference to the detailed theory employed in [7].) We conclude that φ_{mi} maximises all the three quantities $E_\varphi \cdot F_\varphi$, $E_\varphi + F_\varphi$ and $I_\varphi(E) + I_\varphi(F)$ and minimises the uncertainty product $\Delta_\varphi E \cdot \Delta_\varphi F$. Yet, as is well known, projections $E^Q(X)$ and $E^P(Y)$ possess common eigenvectors to eigenvalue 0. (For details and further references, cf [10].) this shows that $I_\varphi(E) + I_\varphi(F)$ is only locally maximal in (27) and can assume the absolute maximum 0. Therefore inequality (6) is seen to be trivial in the present case and (7) still does not represent the optimal (least) bound. Clearly, also the uncertainty product $\Delta_\varphi E \cdot \Delta_\varphi F$ only attains now its local minimum which is $|\mathrm{cov}_\varphi(E, F)| = \frac{1}{4}(1 - a_0^2) \neq 0$ (for $a_0 < 1$), as the absolute minimum for projections is trivially 0.

We conclude this section with some comments on two further examples of position and momentum spectral projections. First, let X, Y be real half-lines, then E, F are known to have no common eigenvectors [10]. Still the numerical range of E, F (the set of all pairs (E_φ, F_φ)) is almost the whole unit square, except the corner points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$; this follows immediately from consideration of the set of states $\{\varphi_q = \exp(\mathrm{i}qP)\varphi,\ \varphi_p = \exp(-\mathrm{i}pQ)\varphi,\ q \in R, p \in R\}$ for some arbitrary fixed unit vector φ. On the other hand, the only stationary points of $I_\varphi(E) + I_\varphi(F)$ allowed by (20) lie on the diagonals of the unit square: $E_\varphi = F_\varphi$ or $E_\varphi = 1 - F_\varphi$. Thus there exist no maximal information states in this example and 0 is the supremum.

In the second example X and Y are assumed to be periodic sets, $X = X + d$, $Y = Y + 2\pi/d$. The corresponding spectral projections have been shown to commute [10]. Accordingly the numerical range is the whole closed unit square with the corner points representing the maximal information situations.

The above two examples denote the extreme cases which one may encounter in the search for maximal information of a pair of simple observables: no maxima at all-all conceivable maxima realised. It is remarkable that position and momentum entail the whole spectrum of possibilities.

4. Everett UR within stochastic quantum mechanics

According to the introduction it appears desirable to have an entropic UR in the form (3) with a state-dependent bound; moreover, for a proper comparison with the standard UR (1) this bound should contain two terms, one measuring the degree of commutativity of the observables A and B in the state φ and the other one measuring their probabilistic dependence and correlation in the state. This viewpoint meets a serious obstacle in the fact that in conventional quantum mechanics non-commuting observables do not possess joint probability distributions—which are needed for an information theoretic formulation of correlation [2]. However, this problem can be and has been solved in the framework of stochastic quantum mechanics. In particular, joint observables for position and momentum have been constructed, non-commutativity being reflected in an irreducible unsharpness of measurement values. In this way one obtains the possibility of generalising the Everett UR,

$$I_\varphi(Q)+I_\varphi(P)\leqslant -\ln(e\pi) \tag{28}$$

where

$$I_\varphi(Q)=\int_{-\infty}^{\infty} \mathrm{d}q|\varphi(q)|^2\ln|\varphi(q)|^2$$

$$I_\varphi(P)=\int_{-\infty}^{\infty} \mathrm{d}p|\tilde\varphi(p)|^2\ln|\tilde\varphi(p)|^2.$$

Let W_0 be a positive trace class operator, $\mathrm{Tr}(W_0)=1$, $U(q,p)=\exp[\mathrm{i}(qP-pQ)]$, $W_{qp}=U(q,p)W_0U(q,p)^{-1}$, and further let $\Gamma=\mathbb{R}^2$ denote the phase space and $\mathscr{B}(\Gamma)$ the Borel sets on Γ. Then the POV measure $A:\mathscr{B}(\Gamma)\rightarrow\mathscr{E}(\mathscr{H})$ defined by

$$Z\rightarrow A(Z)=\frac{1}{2\pi}\int_Z \mathrm{d}q\,\mathrm{d}p\,W_{qp} \tag{29}$$

establishes a positive-definite joint probability functional on Hilbert space $\mathscr{H}$. The marginals are *unsharp* position and momentum observables

$$\begin{aligned} &A(X\times\mathbb{R})=Q_f(X):=(\chi_X*f)(Q)\\ &A(\mathbb{R}\times Y)=P_g(Y):=(\chi_Y*g)(P)\\ &f(q)=\langle q|W|q\rangle \qquad g(p)=\langle p|W|p\rangle. \end{aligned} \tag{30}$$

Grabowski [11] proved the generalised entropic UR

$$I_\varphi(Q_f)+I_\varphi(P_g)\leqslant I_\varphi(A)\leqslant -\ln(2\pi e) \tag{31}$$

where

$$\begin{aligned} &I_\varphi(Q_f)=\int_{-\infty}^{\infty} \mathrm{d}q(f*|\varphi|^2)(q)\ln[(f*|\varphi|^2)(q)]\\ &I_\varphi(P_g)=\int_{-\infty}^{\infty} \mathrm{d}p(g*|\tilde\varphi|^2)(p)\ln[(g*|\tilde\varphi|^2)(p)]\\ &I_\varphi(A)=\frac{1}{2\pi}\int \mathrm{d}q\,\mathrm{d}p\langle\varphi|W_{qp}|\varphi\rangle\ln(\langle\varphi|W_{qp}|\varphi\rangle 2\pi). \end{aligned} \tag{32}$$

The first inequality is due to subadditivity of the entropy functional. The entropic UR (31) has been shown to imply the generalised Heisenberg UR

$$(\Delta_\varphi Q_f)^2(\Delta_\varphi P_g)^2\geqslant 1 \qquad (\hbar=1) \tag{33}$$

with variances defined through the distributions occurring in (32); moreover, for $W_0=P_\varphi$ (33) leads back to the original Heisenberg UR [11]

$$(\Delta_\varphi Q)^2(\Delta_\varphi P)^2\geqslant\tfrac14 \tag{34}$$

since we have

$$(\Delta_\varphi Q_f)^2=(\Delta_\varphi Q)^2+(\Delta f)^2 \qquad (\Delta_\varphi P_g)^2+(\Delta_\varphi P)^2+(\Delta g)^2. \tag{35}$$

Finally, Gaussian states $W_0 = P_\varphi$ lead to equalities in (31) as well as in (33) and (34). Thus maximal information and minimal uncertainty can be achieved and will again coincide. We conclude with a simple consequence of (31). Following Everett [2] we define the correlation information for Q_f, P_g as the expectation of the difference between conditional information $I_\varphi(Q_f|P_g)$ and marginal information $I_\varphi(Q_f)$:

$$\{Q_f, P_g\}_\varphi = \frac{1}{2\pi}\int_\Gamma \mathrm{d}q\,\mathrm{d}p\langle\varphi|W_{qp}|\varphi\rangle \ln\left(\frac{\langle\varphi|W_{qp}|\varphi\rangle/2\pi}{(g*|\tilde{\varphi}|^2)(p)}\right) - I_\varphi(Q_f). \tag{36}$$

It follows from (31) that

$$\{Q_f, P_g\}_\varphi = I_\varphi(A) - I_\varphi(Q_f) - I_\varphi(P_g) \geq 0$$

and finally

$$I_\varphi(Q_f) + I_\varphi(P_g) = I_\varphi(A) - \{Q_f, P_g\}_\varphi \leq -\ln(2\pi e) - \{Q_f, P_g\}_\varphi. \tag{37}$$

Thus the entropic UR for the (unsharp) marginal observables has exactly the desired form: one term strictly less than 0 due to non-commutativity and one term—again negative—accounting for the correlations. This illustrates some aspects of the conceptual power of the generalised POV observable concept in improving on the flexibility of the quantum mechanical formalism.

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Appendix 1

Let E, F be a pair of projections which are not orthogonal to each other, i.e. $E \nleq 1 - F$. Then $\langle\varphi|E|\varphi\rangle\langle\varphi|F|\varphi\rangle$ attains its absolute maximum $\frac{1}{4}\lambda_0^2$ for some $\varphi_0 \in \mathscr{H}$ if and only if $(E+F)\varphi_0 = \lambda_0\varphi_0$ and $\lambda_0 = \|E+F\|$.

Proof. First assume $(E+F)\varphi_0 = \lambda_0\varphi_0$ and $\lambda_0 = \|E+F\|$. Taking the scalar product with $E\varphi_0$ and $F\varphi_0$ yields (setting $\langle\varphi|E|\varphi\rangle = (E)_\varphi$, etc)

$$\lambda_0(E)_{\varphi_0} = (E)_{\varphi_0} + (EF)_{\varphi_0}$$

$$\lambda_0(F)_{\varphi_0} = (F)_{\varphi_0} + (FE)_{\varphi_0}.$$

In particular, $(EF)_{\varphi_0} = (FE)_{\varphi_0}$ must be real and therefore

$$0 = (\lambda_0 - 1)[(E)_{\varphi_0} - (F)_{\varphi_0}].$$

Now $\lambda_0 = \|E+F\| = 1$ is equivalent to $E+F \leq 1$ which is excluded by assumption. This proves that $\lambda_0 = \|E+F\| > 1$ and thus $(E)_{\varphi_0} = (F)_{\varphi_0} = \lambda_0/2$. Consequently, $(E)_\varphi(F)_\varphi \leq \frac{1}{4}\|E+F\|^2 = \frac{1}{4}\lambda_0^2 = (E)_{\varphi_0}(F)_{\varphi_0}$ where the inequality comes from equation (5).

Conversely, the absolute maximum (supremum) of $(E)_\varphi \cdot (F)_\varphi$ must be $\frac{1}{4}\lambda_0^2 = \frac{1}{4}\|E+F\|^2$. For, considering a sequence Ψ_n such that $\|A\Psi_n\| \to 0$ for $A = \|E+F\|I - (E+F) \geq 0$; then $EA\Psi_n \to 0$ and $FA\Psi_n \to 0$ imply $\mathrm{Im}(EF)_{\Psi_n} \to 0$, thus

$(EF)_{\Psi_n} - (FE)_{\Psi_n} \to 0$, so that finally $(EA - FA)_{\Psi_n} \to (\|E+F\| - 1)[(E)_{\Psi_n} - (F)_{\Psi_n}] \to 0$, or $(E)_{\Psi_n}(F)_{\Psi_n} \to (E)^2_{\Psi_n} \to (\frac{1}{2}\|E+F\|)^2$. Now let $(E)_{\varphi_0}(F)_{\varphi_0} = \frac{1}{4}\|E+F\|^2$; then (5) implies $(E+F)_{\varphi_0} = \|E+F\|$ and $\Delta_{\varphi_0}(E+F) = 0$, i.e. $(E+F)\varphi_0 = \|E+F\|\varphi_0$.

Appendix 2

Let E, F be projections such that $E \wedge F = 0$. Then $(E+F)\varphi = \lambda\varphi$ implies $EFEf = (\lambda-1)^2 f$, $FEFg = (\lambda-1)^2 g$ with $\varphi = f + g$, $Ef = f$, $Fg = g$. Conversely, $EFEf = a^2 f$, $Ef = f$ gives rise to a solution of $(E+F)\varphi = \lambda\varphi$ with $\lambda = 1 + a$. The maximal eigenvalue a^2 of EFE leads to the maximal eigenvalue $\lambda = 1 + a$ of $E + F$.

Proof. Any eigenvector φ of $E+F$ is contained in the range $\mathrm{ran}(E+F) \subseteq \mathrm{ran}(E \vee F)$ so that it admits a decomposition $\varphi = f + g$ with $Ef = f$, $Fg = g$. This decomposition is unique since $E \wedge F = 0$. Then one has

$$0 = (E+F)\varphi - \lambda\varphi = [Eg - f(\lambda-1)] + [Ff - g(\lambda-1)].$$

However, $Eg - f(\lambda - 1) \in \mathrm{ran}(F)$ and $Ff - g(\lambda-1) \in \mathrm{ran}(E)$ so that both vectors must vanish due to $E \wedge F = 0$:

$$Eg = (\lambda-1)f =: af \qquad Ff = (\lambda-1)g =: ag \qquad a = \lambda - 1.$$

Case 1. $\lambda = 1$ yields $Eg = 0 = Ff$, i.e. $EFEf = 0 = FEFg$; conversely, $EFEf = 0$ for $Ef = f$ implies $FEf = 0$ and thus $(E+F)f = Ef + Ff = f + FEf = f$, i.e. $\lambda = 1$.

Case 2. $\lambda \neq 1$ gives $f = a^{-1}Eg$, $g = a^{-1}Ff$, $a = \lambda - 1$. From this one calculates $EFEf = EFf = aEg = a^2 f$ and similarly $FEFg = a^2 g$; conversely, $EFEf = a^2 f$ gives $FEF(Ff) = F(EFEf) = a^2 Ff$, $\|Ff\|^2 = \langle f | EFEf\rangle = a^2\|f\|^2 = a^2$. It is easy to show that $a^{-1}F$ induces a unitary map of the spectral space of EFE for the eigenvalue a^2 onto the corresponding spectral space for FEF, the inverse being $a^{-1}E$. Thus, for $g = a^{-1}Ff$ it follows that $a^{-1}Eg = f$ and finally $(E+F)(f+g) = (1+a)(f+g)$.

References

[1] Hirschman I I 1957 *Am. J. Math.* **79** 152
[2] Everett H 1973 *The many-world interpretation of quantum mechanics* ed B S DeWitt and N Graham (Princeton, NJ: Princeton University Press) p 3
[3] Mamojka B 1974 *Int. J. Theor. Phys.* **11** 73
[4] Bialynicki-Birula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
[5] Deutsch D 1983 *Phys. Rev. Lett.* **50** 631
[6] Partovi M H 1983 *Phys. Rev. Lett.* **50** 1883
[7] Lahti P J 1985 *Rep. Math. Phys.* **21** 267
[8] Beckner W 1975 *Ann. Math.* **102** 159
[9] Busch P 1986 *Phys Rev.* D **33** 2253
[10] Busch P and Lahti P J 1986 *Phys. Lett.* **115A** 259
[11] Grabowski M 1984 *Rep. Math. Phys.* **20** 153